**NEW ZEALAND SCHOLARSHIP 2004**

**CALCULUS**

**Sample of assessed candidate work – Performance Descriptor 2: Performance Category 5**

1 (a)

Marker has circled the error – it should be 3.

$$V = \frac{(2x)^2 \times \sqrt{(L-x)^2 - x^2}}{②}$$

Marker code indicates to candidate that a minor error has been made, which has been ignored. The candidate's subsequent working is consistent with the error and there is sufficient evidence to show the candidate is able to devise and/or use models to solve complex problems.

$$V = 2x^2 \times \sqrt{L^2 - 2Lx} \quad \text{MEI}$$

$$= 2x^2(L^2-2Lx)^{\frac{1}{2}}_{g} \qquad f' = 4x \quad g' = -2L(L^2-2Lx)^{-\frac{1}{2}}$$

$$\frac{dv}{dx} = 2x^2 \times -2L(L^2-2Lx)^{-\frac{1}{2}} + 4x(L^2-2Lx)^{\frac{1}{2}}$$

$$= \frac{-4x^2L}{\sqrt{L^2-2Lx}} + 4x\sqrt{(L^2-2Lx)} \qquad \frac{dv}{dx} = 0 \text{ at max/min and inflection point}$$

$$0 = 4x\left(\sqrt{L^2-2Lx} - \frac{xL}{\sqrt{L^2-2Lx}}\right)$$

$$x = 0$$

$$\text{or } \sqrt{L^2-2Lx} = \frac{xL}{\sqrt{L^2-2Lx}}$$

$$L^2 - 2Lx = xL$$

$$L^2 - 3xL = 0$$

$$L(L-3x) = 0$$

$$L - 3x = 0$$

$$L = 3x \qquad x = L/3$$

Double lines used by marker to indicate point at which grade is awarded.

BM awarded.

Turning points at $x = 0$ and $x = L/3$ //

$$\frac{dv}{dx} = -\underset{(f)}{4x^2L}\underset{(g)}{(L^2-2Lx)^{-\frac{1}{2}}} + \underset{(f)}{4x}.\underset{(g)}{(L^2-2Lx)^{\frac{1}{2}}} \quad f' = 4 \qquad g' = L(L^2-2Lx)^{-\frac{1}{2}}$$

$$\left(f' = {}^{-}8xL\right) \quad \left(g' = \frac{+\cancel{2}L}{\cancel{2}}(L^2-2Lx)^{-\frac{3}{2}}\right)$$

$$\frac{d^2v}{d^2x} = 4x^2L^2(L^2-2Lx)^{\frac{3}{2}} - 8xL(L^2-2Lx)^{-\frac{1}{2}} + 4(L^2-2Lx)^{\frac{1}{2}} - 4xL(L^2-2Lx)^{-\frac{1}{2}}$$

when $x = \frac{L}{3}$

*Continued over.*

$$\frac{d^2v}{d^2x} = 4\left(\frac{L}{3}\right)^2 L^2\left(L^2 - 2L\left(\frac{L}{3}\right)\right)^{-3/2} - 8\left(\frac{L}{3}\right)L\left(L^2 - 2L\left(\frac{L}{3}\right)\right)^{-1/2} + 4\left(L^2 - 2L\left(\frac{L}{3}\right)\right)^{1/2} - 4\left(\frac{L}{3}\right)L\left(L^2 - 2L\left(\frac{L}{3}\right)\right)^{-1/2}$$

$$= \frac{4L^4}{9}\left(L^2 - \frac{2L^2}{3}\right)^{-3/2} - \frac{8L^2}{3}\left(L^2 - \frac{2L^2}{3}\right)^{-1/2} + 4\left(L^2 - \frac{2L^2}{3}\right)^{1/2} - \frac{4L^2}{3}\left(L^2 - \frac{2L^2}{3}\right)^{-1/2}$$

BM has already been achieved. Candidate has differentiated again to prove a maximum was not required – this work was unnecessary.

$$= \frac{4L^4}{9}\sqrt{\left(\frac{L^2}{3}\right)^{-3/2}} - \frac{8L^2}{3} \Big/ \sqrt{\frac{L^2}{3}} + 4\sqrt{\frac{L^2}{3}} - \frac{4L^2}{3} \Big/ \sqrt{\frac{L^2}{3}}$$

$$= \frac{4L^4}{9} \Big/ \sqrt{\frac{L^{-5}}{27}} - \frac{8L^2}{3} \Big/ \frac{L}{\sqrt{3}} + 4\frac{L}{\sqrt{3}} - \frac{4L^2}{3} \Big/ \frac{L}{\sqrt{3}}$$

$$= \frac{4L^4}{9} \Big/ \frac{L^3}{\sqrt{27}} - \frac{8L^2\sqrt{3}}{3L} + \frac{4L}{\sqrt{3}} - \frac{4L^2\sqrt{3}}{3L} = \frac{4}{9}L^{12}$$

$$= \frac{4L^4\sqrt{27}}{9L^3} - \frac{8L\sqrt{3}}{3} + \frac{4L}{\sqrt{3}} - \frac{4L\sqrt{3}}{3}$$

$$= \frac{4L\sqrt{27}}{9} - \frac{12L\sqrt{3}}{3} + \frac{4L}{\sqrt{3}}$$

$$= L\left(\frac{4\sqrt{27}}{9} - \frac{12\sqrt{3}}{3} + \frac{4}{\sqrt{3}}\right)$$

= $-2.3094L$ and L must be positive so $\frac{dv^2}{d^2x} = -2.3094L$ when $x = L/3$

N – no credit given.

$\therefore$ maximum turning point when $x = L/3$ //

The marker has indicated this work is incorrect. It was also unnecessary – the candidate had already met the requirements for BM grade.

1 (b)

$$\frac{dv}{dt} = \frac{k^2}{3L} \qquad \frac{dh}{dt} > \frac{L-2k}{h+1}$$

$$V = \frac{\left(2^2 k\right)^2 h}{2}$$

$$V = 2k^2 h$$

$$\frac{dv}{dh} = 2k^2$$

$$\frac{dh}{dv} = \frac{1}{2k^2} \qquad \frac{dh}{dt} = \frac{dh}{dv} \times \frac{dv}{dt}$$

$$= \frac{k^2}{6k^2 L} = 1/6L$$

$$\frac{L-2k}{h+1} < \frac{1}{6L}$$

$$6L^2 - 12kL < h + 1$$

$$6L^2 - 12kL - h - 1 < 0$$

$$L^2 - 2kL - \frac{h+1}{6}$$

$$(L-k)^2 - k^2 - \frac{h+1}{6} < 0$$

$$L - k < \sqrt{\frac{h}{6} + \frac{1}{6} + k^2}$$

$$L < k \pm \sqrt{\frac{h+1}{6} + k^2} \quad //$$

Not correct.

2 (a) (i)

Tangent of parabola $= y^2 = 4ax$

$$2y\frac{dy}{dx} = 4a$$

$$\frac{dy}{dx} = \frac{2a}{y} \quad \text{when } y = \text{P}$$

$$\frac{dy}{dx} = 2a/\text{P} \qquad y - \text{P} = \frac{2a}{\text{P}}(x - R)$$

$$y = \frac{2ax}{\text{P}} - \frac{2aR}{\text{P}} + \text{P}$$

Tangent of ellipse $= \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0 \qquad \frac{dy}{dx} = \frac{-\not{2}xb^2}{\not{2}ya^2} \quad \text{and } y = \text{P} \quad x = R$$

$$y - \text{P} = \frac{-Rb^2}{\text{P}a^2}(x - R)$$

$$y = \frac{-xRb^2}{\text{P}a^2} + \frac{b^2R^2}{\text{P}a^2} + \text{P}$$

$$\frac{-xRb^2}{\text{P}a^2} + \frac{xb^2R^2}{\text{P}a^2} + \not{\text{P}} = \frac{2a^2}{\text{P}} - \frac{2aR}{\text{P}} + \not{\text{P}}$$

$$\frac{-xRb^2}{a^2} + \frac{b^2R^2}{a^2} = 2ax - 2aR$$

$$2a^3x + xRb^2 = b^2R^2 + 2a^3R \quad //$$

Candidate 'fudged' the answer. N awarded (not correct).

P = $y$ coordinate of P
$R$ = $x$ coordinate of P

$$2y\frac{dy}{dx} = 4a \quad \frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{2a}{y} \quad \frac{dy}{dx} = \frac{b^2}{2y} - \frac{2xb^2}{a^2y} \quad = \frac{b^2}{2y} - \frac{xb^2}{ya^2}$$

$$= \frac{2a}{\text{P}} \qquad\qquad = \frac{b^2}{2\text{P}} - \frac{Rb^2}{\text{P}a^2}$$

to be perpendicular one must = inverse negative the other

*Continued over.*

$$\frac{b^2}{2P} - \frac{Rb^2}{Pa^2} = \frac{-P}{2a}$$

$$\frac{1}{2} - \frac{R}{a^2} = \frac{P^2}{2ab^2}$$

Error circled – sign missing.

NC

Marker code indicates that the candidate has made errors and subsequent working is not consistent with those errors.

$$P^2 = 4aR \times \frac{1}{2} - \frac{R}{a^2} = \frac{4aR}{2ab^2}$$

$$\frac{1}{2} - \frac{R}{a^2} = \frac{2R}{b^2}$$

$$\frac{1}{2R} - \frac{1}{a^2} = \frac{-2}{b^2}$$

$$\frac{1}{2R} = \frac{-2}{b^2} + \frac{1}{a^2} \qquad \frac{-2a^2 + b^2}{a^2b^2} = \frac{1}{2R}$$

$$\frac{-2a^2 + b^2}{a^2b^2} = 0$$

Error circled, $\frac{1}{2R}$ became 0.

$$-2a^2 + b^2 = 0$$

$$b^2 = 2a^2 \quad //$$

RAWW.

The marker code tells the candidate that this is the right answer but the working is incorrect; it was 'fudged' at the end.

2 (a) (ii)

$$\sqrt{R_1^2 P^2}$$

$$\frac{r^2}{\frac{b^2}{2}} + \frac{P^2}{b^2} = 1 \qquad \frac{2r^2 + P^2}{b^2} = 1$$

$$2r^2 + P^2 = b^2$$

$$r^2 = \frac{b^2}{2} - \frac{P^2}{2} \qquad P^2 = b^2 - 2r^2$$

$$r^2 + P^2 = \frac{b^2}{2} - \frac{P^2}{2} + b^2 - 2r^2$$

$$\sqrt{r^2 + P^2} = \sqrt{\frac{3b^2}{2} - \frac{P^2}{2} - 2r^2}$$

$$\overline{PO} = \sqrt{3a^2 - \frac{P^2}{2} - 2r^2} \quad //$$

N – not correct.

3. (a) (i)

$y = \dfrac{x^2}{1+x^2}$ $\quad \begin{matrix} f \\ g \end{matrix}$ $\quad \begin{matrix} f' = 2x \\ g' = 2x \end{matrix}$ $\quad \dfrac{dy}{dx} = \dfrac{(1+x^2)2x - 2x^3}{(1+x^2)^2} = \tfrac{1}{2}$

$\dfrac{dy}{dx} = \dfrac{2x}{(x^2+1)^2}$ $\qquad \dfrac{2x}{(1+x^2)} - \dfrac{2x^3}{(1+x^2)^2} = \tfrac{1}{2}$

$$2x(1+x^2) - 2x^3 = \frac{(1+x^2)^2}{2}$$

$$2x + 2x^3 - 2x^3 = \frac{x^4 + 2x^2 + 1}{2}$$

$4x = x^4 + 2x^2 + 1$

$x^4 + 2x^2 - 4x + 1 = 0$

$(x–1)$ is a factor

$(x-1)\left(x^3 + x^2 + 3x - 1\right)$

$$\begin{array}{r} x^3 + x^2 + 3x - 1 \\ x - 1 \overline{\big) x^4 + 0x^3 + 2x^2 - 4x + 1} \\ \underline{x^3 \; - x^3} \qquad\qquad\qquad \\ + \quad x^3 + 2x^2 \qquad\quad \\ \underline{x^3 - x^3} \qquad\quad\; \\ + 3x^2 - 4x \quad\; \\ \underline{3x^2 - 3x} \quad\; \\ -x + 1 \\ \underline{-x - 1} \end{array}$$

$f(x) = x^4 + 2x^2 - 4x + 1$

$f'(x) = 4x^3 + 4x - 4$

$$x_n = x_6 - \frac{f(x)}{f'(x)}$$

$$= \tfrac{1}{4} - \frac{(\frac14)^4 + 2(\frac14)^2 - 4(\frac14) + 1}{4(\frac14)^3 + 4(\frac14) - 4}$$

$x_1 = .2939$
$x_2 = .2956$
$x_3 = .2956$ (4sf)

Candidate used a method not covered by Level 3 Achievement Standards. Solved the problem, BM awarded.

Point $x$ = .2956 has gradient of ½ as well as $x$ = 1

Check $\dfrac{2(.2956)}{(.2956^2+1)^2} \approx 0$ //

3 (a) (ii)

$y = \frac{x^2}{1+x^2}$

$\frac{1}{y} = \frac{1+x^2}{x^2}$ $\quad V = \pi\int_0^{½} \frac{y}{1-y}.dy \quad \frac{f}{g}$

$\frac{1}{y} = \frac{1}{x^2} + 1 \qquad \frac{1}{y} - 1 = \frac{1}{x^2}$

$x^2 = \frac{1}{\frac{1}{y}-1} = \frac{y}{1-y}$

$\frac{y}{1-y} \quad \frac{1-u}{u}$

$= \frac{y+y^2}{1-y^2} \quad \frac{1}{u} - 1 = \frac{1}{1-y} - 1 - x^2$

$x^2 = \frac{1}{1-y} - 1 \quad v = \pi\int_0^{½} \frac{1}{1-y} - 1 \,.\, dy$

$= \pi\left[{}^{-}ln(1-y) - y\right]_0^{½}$

$= \pi\left[\left({}^{-}ln\left(1-\frac{1}{2}\right) - ½\right) - \left({}^{-}ln(1-0) - 0\right)\right]$

$= \pi\left[.19315 - 0\right]$

$= .19315\,\pi$

$= .6068$ units$^3$ //

BM grade awarded.

3 (b) (i)

$\int_0^a f(x).dx = \int f(a) - \int f(0)$

$\int_0^a f(a-x).dx = \int f(a-a) - \int f(a)$

$= \int f(0) - \int f(a)$

Marker has circled an error.

$\therefore \int_0^a f(x).dx = \int_0^a f(x).dx$

and both equal $\int f(0) - \int f(a)$

say $f(x) = Ax^2 + Bx + c$ $\quad f(a+x) = A(a-x)^2 + B(a-x) + c$

$\int_0^a Ax^2 + Bx + c.dx$

$= \left[\frac{A}{3}x^3 + \frac{B}{2}x^2 + cx\right]_0^a$

$= \frac{A}{3}a^3 + \frac{B}{2}a^2 + ca$

$\int_0^a A(a-x)^2 + B(a-x) + c.dx$

$= \left[\frac{-A}{3}(a-x)^3 - \frac{B}{2}(a-x)^2 + cx\right]_0^a$

$= \left[\frac{-A}{3}(a-a)^3 - \frac{B}{2}(a-a)^2 + ca - \left(\frac{-A}{3}(a)^3 - \frac{B}{2}(a)^2\right) + ca\right]$

$= \frac{A}{3}a^3 + \frac{B}{2}a^2 + ca$

so $\int_0^a f(x).dx = \int_0^a f(a-x).dx$

N – not correct.

3 (b) (ii)

$\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + (o)^n x}.dx$

if $n$ = even, then $\sin^n x + \cos^n x = 1$

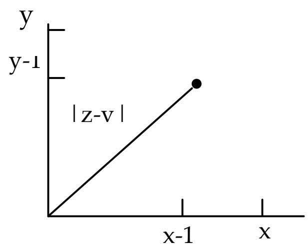
so $\int_0^{\frac{\pi}{2}} \sin^n x.dc$ $\quad$ here $n$ = even

$(\sin x)^n$

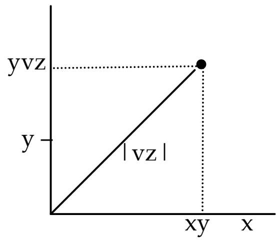
N – not correct.

4 (a) (i)

$$|z-v| = |x+iy-1-i| = |(x-1)+i(y-1)|$$
$$= \sqrt{(x-1)^2+(y-1)^2}$$




$$|vz| = |(1+i)(x+iy)|$$
$$= |x-y+iy+ix|$$
$$= |(x-y)+i(y+x)|$$
$$= \sqrt{(x-y)^2+(y+x)^2}$$




$$\sqrt{(x-1)^2+(y-1)^2} = \sqrt{(x-y)^2+(y+x)^2}$$

$$x^2-2x+1+y^2-2y+1 = x^2-2xy+y^2+y^2+x^2+2xy$$
$$x^2-2x+1+y^2-2y+1 = 2x^2+2y^2$$
$$2x^2+2y^2-x^2+2x-1-y^2+2y-1=0$$
$$x^2+2x+y^2+2y-2=0$$
$$(x+1)^2+(y+1)^2-4=0$$
$$(x+1)^2+(y+1)^2=4$$

has centre $x=-1$ $y=-1$ $(-1,-1)$
and radius of $\sqrt{4}=2$ //

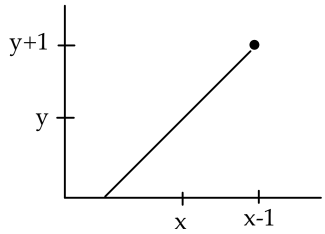
BS grade awarded.

4 (a) (ii)

$|z-v| = \sqrt{(x-1)^2 + (y-1)^2}$ (from part i)

$|z+v| = |1+i+x+iy| = |(x+1)+i(y+1)|$

$= \sqrt{(x+1)^2 + (y+1)^2}$




$|z-v| = |z+v| \quad \therefore \sqrt{(x-1)^2 + (y-1)^2} = \sqrt{(x+1)^2 + (y+1)^2}$

$x^2 - 2x + 1 + y^2 - 2y + 1 = x^2 + 2x + 1 + y^2 + 2y + 1$

$-2y - 2y = 2y + 2y$

$-4y = 4x$

$y = -x$ intersects with $y = -x$ & $(x+1)^2 + (y+1)^2 = 4$

so $x^2 + 2x + 1 + (-x+1)^2 = 4$

$x^2 + 2x + 1 + x^2 - 2x + 1 = 4$

$2x^2 + 2 = 4$

$x^2 = 1$ (MEI)

$x = 1 \quad y = -x = -(1)y \quad y = -1$

so a circle from part (i) and line $|z-v| = |z+v|$

intersect with $x = 1$ & $y = -1$

at point $(1, -1)$ //

Marker code indicates to candidate that a minor error has been made and ignored. It has not prevented the candidate demonstrating an ability to select appropriate combinations of techniques and concepts to solve complex problems.

BS grade awarded.

5. *[Candidate did not attempt Question 5.]*

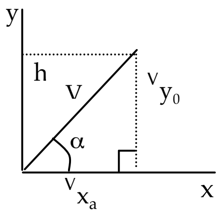
6 (a)

$$\frac{dy}{dt} = \int -y.dt = -gt + c$$

$$dy = \int -gt + c.dt \qquad y = -gt^2 + c_1 t + c_2 \quad \text{when } t = 0\ y = 0$$

$$\text{so } 0 = c_2 \qquad \text{so } c_2 = 0$$

$$\therefore\ y = -gt^2 + c_1 t$$




vertical velocity $= \sin\alpha = \dfrac{Vy}{v}$

$$V\sin\alpha = Vy \qquad Vy = \frac{dy}{dt}$$

horizontal velocity $= \cos\alpha = \dfrac{Vx}{V}$

$$V\cos\alpha = Vx \qquad Vx = \frac{dx}{dt}$$

$$\frac{dy}{dt} = -gt + c = V\sin\alpha$$

$$c = V\sin\alpha + gt$$

$$\therefore\ y = -gt^2 + (V\sin\alpha + gt)t$$

$$y = -gt^2 + Vt\sin\alpha + gt^2$$

$$y = Vt\sin\alpha$$

$$\frac{dx}{dt} = \int 0\ dt = C$$

$$x = \int c.dt = Ct + C_2 \quad \text{when } x = 0\ \&\ t = 0$$

$$\text{so } 0 = C_2 \quad C_2 = 0$$

$$\text{so } x = Ct$$

$$\frac{dx}{dt} = C = V\cos\alpha \qquad \text{so } C = V\cos\alpha$$

Errors circled by marker – incorrect.

$$\text{so } x = V\cos\alpha + t$$

$$t = \frac{x}{V\cos\alpha}$$

$$\cancel{y} = -\cancel{g}\left(\frac{\cancel{x}}{\cancel{V\cos\alpha}}\right)^2 + \left(\frac{\cancel{x}}{\cancel{V\sin\alpha}\,\cancel{g}(\cancel{V\cos\alpha})}\right) \times \frac{\cancel{x}}{\cancel{V\cos\alpha}}$$

$$y = -\cancel{X}\left(\frac{\cancel{X}}{\cancel{X}\cancel{xxx}\cancel{x}}\right)^2 + V\left(\frac{x}{V\cos\alpha}\right)\sin\alpha + \cancel{X}\left(\frac{\cancel{X}}{\cancel{X}\cancel{xxx}\cancel{x}}\right)^2$$

$$= -\cancel{g}\frac{\cancel{x}^2}{\cancel{V}^2\cancel{\cos}^2\cancel{\alpha}} + \frac{\cancel{gx}^2V\cancel{\sin\alpha}}{\cancel{V}^2\cancel{\cos}^2\cancel{\alpha}}$$

$$y = \frac{\cancel{gx}^2}{\cancel{V}^2\cancel{\cos}^2\cancel{\alpha}}\left(\cancel{V}\cancel{\sin\alpha} - \cancel{1}\right)$$

$$= -g\frac{x^2}{V^2\cos^2\alpha} + \frac{x\sin\alpha}{\cos\alpha} + \frac{gx^2}{V^2\cos^2\alpha}$$

$$y = \frac{x\sin\alpha}{\cos\alpha}$$

N – not correct.

*Continued over.*

6 (a) continued

$\frac{d^2x}{dt^2} = 0 \quad \frac{dx}{dt} = \int 0.dt = c_1 \qquad \& \ x = \int C.dt = C_1 t + C_2$

when $t = 0 \ x = 0$

so $0 = 0 + C_2 \qquad C_2 = 0$

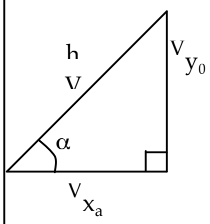
so $x = Ct$

$\frac{d^2y}{dt^2} = -g \quad \frac{dy}{dt} = \int -g.dt = -gt + C_1 \qquad y = -gt^2 + C_1 t + C_2$

when $t = 0 \ y = 0$

so $0 = 0 + 0 + C_2 \qquad C_2 = 0$

so $y = -gt^2 + ct$




$\cos\alpha = \frac{Vx}{V} \qquad Vx = V\cos\alpha \qquad Vx = \frac{dx}{dt}$

$\sin\alpha = \frac{Vy}{V} \qquad Vy = V\sin\alpha \qquad Vy = \frac{dy}{dt}$

when $t = 0 \ \frac{dx}{dt} = C_1$ so $C = V\cos\alpha$

so $x = Vt\cos\alpha \qquad t = \frac{x}{V\cos\alpha}$

when $t = 0 \ \frac{dy}{dt} = C_1$ so $C = V\sin\alpha$

so $y = -gt^2 + Vt\sin\alpha$ MEI

Error circled by marker. Marker code indicates it was a minor error, which was ignored.

so $y = \frac{-gx^2}{V^2\cos^2\alpha} + \frac{Vx\sin\alpha}{V\cos\alpha}$

$y = \frac{-gx^2}{V^2\cos^2\alpha} + \frac{x\sin\alpha}{\cos\alpha}$

$= \frac{-gx^2}{V^2\cos^2\alpha} + x\tan\alpha$ //

Candidate made a minor error, which was ignored. Sufficient evidence to show candidate can devise and/or use models to solve complex problems – BM awarded.

6 (b) (i)

$$y = \frac{-gx^2}{V^2\cos^2\alpha} + x\tan\alpha$$

$$h = \frac{-gk^2h^2}{V^2\cos^2\alpha} + kh\tan\alpha$$

$$\frac{-gk^2h}{\left(1+\sqrt{1+k^2}\right)\cos^2\alpha} + kh\tan\alpha = h$$

$$\frac{-k^2h}{1+\sqrt{1+k^2}} + kh\tan\alpha - h = 0$$

$$h\left(-1 + k\tan\alpha - \frac{k^2}{1+\sqrt{1+k^2}\cos^2\alpha}\right) = 0$$

$$h\left(-1 + k\tan\alpha\frac{-k^2}{\left(1+\sqrt{1+k^2}\right)\cos^2\alpha}\right) = 0$$

$$-1 + k\tan\alpha - \frac{k^2}{\left(1+\sqrt{1+k^2\cos^2\alpha}\right)} = 0$$

$$\frac{\sin\alpha}{\cos\alpha} - \frac{k}{\left(1+\sqrt{1+k^2}\right)\cos^2\alpha} = \frac{1}{k}$$

$$\sin\alpha - \frac{k}{\left(1+\sqrt{1+k^2}\right)\cos\alpha} = \frac{\cos\alpha}{k} \quad //$$

Not correct.

**General Comments**

Candidate met prerequisite for overall achievement of Performance Descriptor 2. Total of 6 Bs awarded falls in Performance Category 5.

# NEW ZEALAND SCHOLARSHIP 2004

# CALCULUS

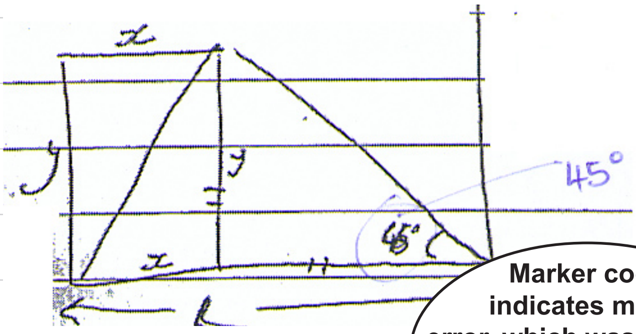
## Sample of assessed candidate work – Performance Descriptor 2 – Performance Category 3

Assessor's use only

Question number

1. Value of pyramid $= \frac{1}{3}$ base $\times h$

a) $h = \sqrt{y^2} - x^2 \qquad b = 4x^2$




$y = L - x$

$h = ((L-x)^2 - x^2)^{\frac{1}{2}}$

Value of pyramid $= \frac{1}{3} \times 4x^2 \times (L^2 - 2Lx + x^2 - x^2)^{\frac{1}{2}}$

$= 4x^2 \times (L^2 - 2Lx)^{\frac{1}{2}}$

MEI

Marker code indicates minor error, which was ignored. Minor error **did not** simplify the problem/ working.

$L$ is constant so

$$\frac{dv}{da} = \frac{8x(L^2 - 2Lx)^{\frac{1}{2}} + 2x^2 \times (L^2 - 2Lx)^{-\frac{1}{2}} \times -2L}{3}$$

$$= \frac{8x(L^2 - 2Lx)^{\frac{1}{2}} - 4x^1 L (L^2 - 2Lx)^{-\frac{1}{2}}}{3}$$

max/min when $\frac{dv}{dx} = 0 \Rightarrow 4x\left((L^2 - 2Lx)^{\frac{1}{2}} - xL\,(L^2 2Lx)^{-\frac{1}{2}}\right) = 0$

$\Rightarrow \sqrt{L^2 - 2Lx} = \frac{xL}{\sqrt{L^2 - 2Lx}}$

$L^2 - 2Lx = xL$

$L^2 = xL + 2Lx$

$L^2 = x(L + 2L)$

$\frac{L^2}{3L} = x$

$\Rightarrow x = \frac{L}{3}$

max pyramid area when $x = \frac{1}{3}L$

CON

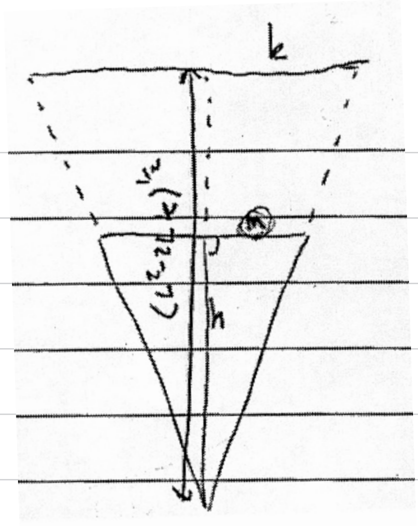
Marker code 'C' indicates incorrect answer, but it is consistent with the error hence BM awarded.

**BM**

Assessor's use only

Question number

1. $\frac{dv}{dt} = \frac{k^2}{3L}$ $\quad V = \frac{1}{3} b \times h$




b) hmmm... $\quad$ need $\frac{dv}{dh}$

$V \text{ total} = \frac{1}{3} \times 4k^2 (L^2 - 2Lk)^{\frac{1}{2}}$

$\text{total } H = (L^2 - 2Lk)^{\frac{1}{2}}$

$=> \frac{k}{\sqrt{L^2 - 2Lk}} = \frac{m}{h} => \frac{hk}{\sqrt{L^2 - 2Lk}}$

$V \text{ water} = \frac{1}{3} b \times h \qquad b = 4m^2 = 4x\left(\frac{hk}{\sqrt{L^2 - 2Lk}}\right)^2 = \frac{4h^2k^2}{L^2 - 2Lk}$

$h = h \qquad\qquad = \frac{4h^2k^2}{L(L - 2k)}$

✓ L² – 2Lk

$V \text{ water} = \frac{1}{3} \times h \times \frac{4h^2k^2}{L(L - 2k)}$

$= \frac{4h^3k^2}{3L(L - 2k)} \qquad L \& k$ are constant

$\frac{dv}{dh} = \frac{12h^2k^2}{3L(L - 2k)} = \frac{4h^2k^2}{L(L - 2k)}$

$\frac{dv}{db} = \frac{dv}{dt} \div \frac{dv}{dh} = \frac{k^2}{3L} \div \frac{4h^2k^2}{L(L - 2k)} = \frac{\cancel{k^2}\, \cancel{L}\, (L - 2k)}{3\cancel{L} \times 4h^2 \cancel{k^2}}$

$= \frac{L - 2k}{12h^2}$

$\frac{L - 2k}{12h^2} > \frac{L - 2k}{h + 1}$

$=> 12h^2 < h + 1 \qquad 12h^2 - h - 1 < 0$

$12h^2 - h - 1 = 0 \quad \text{when } h = \frac{1}{3}, -\frac{1}{4}$

$=> h < \frac{1}{3}$

$=> \text{for } h < \frac{1}{3} \qquad \frac{dh}{dt} > \frac{L - 2k}{h + 1}$

BS

~~AR~~

**Where candidates do not meet the prerequisite for an overall grade of A, any A grades awarded for individual questions provide evidence for B descriptors.**

Assessor's use only

Question number

2.

a) i)

point p:

$$y^2 = 4ax \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

**Error here.**
**$y^2 = 4ax$, not $(4ax)^2$.**
**This simplifies the problem. The candidate does not need to solve a quadratic equation due to the error, therefore it is not considered to be a minor error.**

$$=> \frac{x^2}{a^2} + \frac{(4ax)^2}{b^2} = 1$$

$$=> \frac{x^2}{a^2} + \frac{16a^2x^2}{b^2} = 1$$

$$=> b^2x^2 + 16a^4x^2 = a^2b^2$$

$$=> x^2(b^2 + 16a^4) = a^2b^2$$

$$=> x^2 = \frac{a^2b^2}{b^2 + 16a^4}$$

$$=> x = \frac{ab}{\sqrt{b^2 + 16a^4}}$$

$$y^2 = 4ax$$

$$= \frac{4a^2b^2}{\sqrt{b^2 + 16a^4}}$$

$$=> y = \frac{2a\sqrt{b}}{\sqrt[4]{b^2 + 16a^4}}$$

parabola: $(x, y_1)$

$$y_p^2 = 4ax$$

$$=> 2y_p\frac{dy_p}{dx} = 4a$$

$$=> \frac{dy_p}{dx} = \frac{2a}{y_p}$$

ellipse: $(x, y_e)$

$$\frac{x^2}{a^2} + \frac{y_e{}^2}{b^2} = 1$$

$$=> \frac{2x}{a^2} + \frac{2y_p}{b^2}\frac{dy_e}{dx} = 0$$

$$=> \frac{y_e}{b^2}\frac{dy_e}{dx} = \frac{-x}{a^2}$$

$$=> \frac{dy_e}{dx} = \frac{-b^2x}{a^2y_e}$$

Assessor's use only

Question number

But $\frac{dy_p}{dx}\ \frac{dy_e}{dx} = -1$ at P

$=> \frac{2a}{y} \times \frac{-b^2 x}{a^2 y} = -1$

$=> \frac{2ab^2 x}{a^2 y^2} = 1$

$=> \frac{2b^2 x}{ay^2} = 1$

$=> 2b^2 x = ay^2$

Assessor's use only

Question number

at p:

$$\frac{-2ab^3}{\sqrt{b^2+16a^4}} = \frac{4a^3b}{\sqrt{b^2+16a^4}}$$

$$=> 2ab^3 = 4a^3b$$

$$=> b^2 = 2a^2 \quad \text{as required}$$

**Consistent working – error does not affect result.**

BS

2 (a) (ii) distance $0p$

$$d = \sqrt{x^2+y^2}$$

$$= \sqrt{\frac{a^2b^2}{b^2+16a^4} + \frac{4a^2b}{\sqrt{b^2+16a^4}}}$$

$$= \sqrt{\frac{2a^4}{2a^2+16a^4} + \frac{4\sqrt{2}\,a^3}{\sqrt{2a^2+16a^4}}}$$

$$= \sqrt{\frac{2a^4+4\sqrt{2}\,a^3\sqrt{2a^2+16a^4}}{2a^2+16a^4}}$$

$$= \sqrt{\frac{2a^4+8a^3\sqrt{a^2+8a^4}}{2a^2+16a^4}}$$

$$= \sqrt{\frac{a^2+4a\sqrt{a^2+8a^4}}{1+8a^2}}$$

$$= \sqrt{\frac{a^2+4a^2\sqrt{1+8a^2}}{1+8a^2}}$$

$$= a\sqrt{\frac{1+4\sqrt{1+8a^2}}{1+8a^2}}$$

**Incorrect expressions for $x^2$ and $y^2$ from error in 2(a) (i)**

**Error from 2 (a) (i) caused major error here that simplified the problem**

N

Assessor's use only

Question number

2 (b)

$$b^2 = 2a^2$$
$$=> b = \sqrt{2a}$$

point P:

$$y_p = \frac{2a\sqrt{b}}{\sqrt[4]{b^2 + 16a^4}}$$
$$= \frac{2\sqrt[4]{2}\,a\sqrt{a}}{\sqrt[4]{2a^2 + 16a^4}}$$
$$= \frac{2a}{\sqrt[4]{1 + 8a^2}}$$

**Error from 2 (a) (i) caused multiple problems for the candidate in Question 2 (b)**

$$x_p = \frac{ab}{\sqrt{b^2 + 16a^4}}$$
$$= \frac{\sqrt{2}\,a^2}{\sqrt{2a^2 + 16a^4}}$$
$$= \frac{a}{\sqrt{1 + 8a^2}}$$

$$y = \frac{2a}{y_p}(x - x_p) + y_p \qquad x = 0$$
$$= \frac{-2ax_p}{y_p} + y_p \qquad =$$

Question number

Assessor's use only

tangent to parabola:

$$-y \quad \frac{-2a}{\sqrt[4]{1+8a^2}} = \frac{2a}{y_p}\left(x - \frac{a}{\sqrt{1+8a^2}}\right)$$
$$= \sqrt[4]{1+8a^2}\left(x - \frac{a}{\sqrt{1+8a^2}}\right)$$

$x =$ intercept when $x = 0$

$$=> y = \sqrt[4]{1+8a^2}\left(\frac{-a}{\sqrt{1+8a^2}}\right) + \frac{2a}{\sqrt[4]{1+8a^2}}$$
$$= \frac{-a}{\sqrt[4]{1+8a^2}} + \frac{2a}{\sqrt[4]{1+8a^2}}$$
$$= \frac{a}{\sqrt[4]{1+8a^2}}$$

tangent to ellipse:

$$y = \frac{-b^2 x_p}{a^2 \quad y_p}(x - x_p) + y_p$$
$$= \frac{-2a^2 xp}{a^2 y_p}(x - x_p) + y_p$$
$$= \frac{-2x_p\,(x - x_p)}{y_p} + y_p$$

$x -$ intercept when $x = 0$

$$\therefore y_n = \frac{2{x_p}^2}{y_p} + y_p$$

Question number

$$=> y_N = \frac{2a^2 \sqrt[4]{1+8a^2}}{(1+8a^2)2a} + \frac{2a}{\sqrt[4]{1+8a^2}}$$

$$= \frac{a\sqrt[4]{1+8a^2}}{1+8a^2} + \frac{2a}{\sqrt[4]{1+8a^2}}$$

MN = ($x$ − intercept of ellipse) − ($x$ − intercept of parabola)

$$MN = \frac{a^2\sqrt[4]{1+8a^2}}{1+8a^2} + \frac{2a}{\sqrt[4]{1+8a^2}} - \frac{a}{\sqrt[4]{1+8a^2}}$$

$$= \frac{a\sqrt[4]{1+8a^2}}{1+8a^2} + \frac{a}{\sqrt[4]{1+8a^2}}$$

$$= \frac{a}{(1+8a^2)^{3/4}} + \frac{a}{(1+8a^2)^{1/4}}$$

//

Assessor's use only

Question number

$$y_m = \frac{-2_a x_p}{y_p} + y_p$$

$$= \frac{-2a^2 \sqrt[4]{1+8a^2}}{\sqrt{1+8a^2}\ 2a} + \frac{2a}{\sqrt[4]{1+8a^2}}$$

$$= \frac{-a \sqrt[4]{1+8a^2}}{\sqrt{1+8a^2}} + \frac{2a}{\sqrt[4]{1+8a^2}}$$

$$= \frac{-a}{\sqrt[4]{1+8a^2}} + \frac{2a}{\sqrt[4]{1+8a^2}}$$

$$= \frac{-a}{\sqrt[4]{1+8a^2}}$$

**NS**

**The evidence presented by the candidate is correct, but not enough has been done to meet the requirements of the descriptor, hence a code of NS has been written (not sufficient)**

Assessor's use only

| Question number | |
|---|---|
| 3. a. i) | $\frac{dy}{dx} = \frac{(1+x^2)2x - 2x\,x^2}{(1+x^2)^2}$ |
| | $= \frac{2x + 2x^4 - 2x^3}{(1+x^2)^2}$ |

MEI

**Error in the second line made the problem harder! Candidate still demonstrated they had met the standard – BM awarded. Continued working is consistent with the error – indicated by marker code CON.**

$\frac{dy}{dx} = 1/2$

$=> \frac{2x + 2x^4 - 2x^3}{(1+x^2)^2} = 1/2$

CON

$=> \frac{4x^4 - 4x^3 + 4x}{(x^2+1)^2} = 1$

$=> 4x^4 - 4x^3 + 4x = x^4 + 2x^2 + 1$

$=> 3x^4 - 4x^3 - 2x^2 + 4x - 1 = 0$

$x = 1$ is a root

$$\begin{array}{r} 3x^3 - x^2 - 3x + 1 \\ x - 1 \overline{\big)\, 3x^4 - 4x^3 - 2x^2 + 4x - 1} \\ \underline{3x^4 - 3x^3} \\ -x^3 - 2x^2 \\ \underline{-x^3 + x^2} \\ -3x^2 + 4x \\ \underline{-3x^2 + 3x} \\ x - 1 \\ \underline{x - 1} \end{array}$$

CON

Assessor's use only

Question number

$=> 3x^3 - x^2 - 3x + 1 = 0$

*for remaining 3 roots*

$=> x^2(3x-1) - (3x-1) = 0$

$=> (3x-1)(x^2-1) = 0$

$=> (3x-1)(x-1)(x+1) = 0$

$\therefore x = \frac{1}{3}$ *or* $1$ *or* $-1$

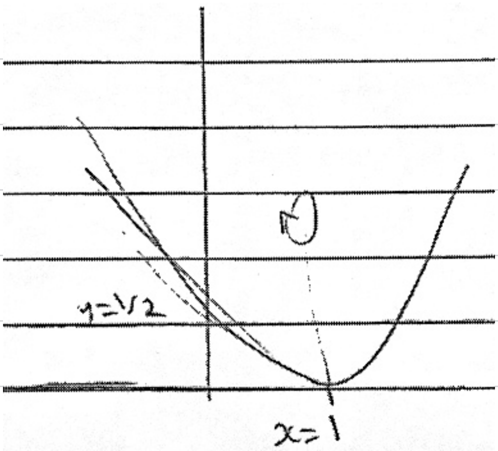
CON

BM

*i.e. gradient* $= \frac{1}{2}$ *at* $x = \frac{1}{3}$

(as required, $\frac{1}{4} < \frac{1}{3} < \frac{1}{2}$)

ii)




*equivalent to rotating area between*

$y = \frac{x^2}{1+x^2}$ and $y = \frac{1}{2}$ around $x = 0$

$y = \frac{x^2}{1+x^2}$

$=> y + x^2y = x^2$

$=> x^2 - x^2y = y$

$=> x^2(1-y) = y$

Assessor's use only

Question number

$$=> x^2 = \frac{y}{1-y}$$

$$V = \pi \int_0^{1/2} x^2 \, dy$$
$$= \pi \int_0^{1/2} \frac{y}{1-y} \, dy$$

substitute $u = 1 - y$

$$=> y = 1 - u$$
$$=> \frac{dy}{du} = -1$$
$$=> dy = -du$$

$$=> V = \pi \int_1^{1/2} \frac{1-u}{u}(-du)$$
$$= \pi \int_1^{1/2} \frac{u-1}{u} du$$
$$= \pi \int_1^{1/2} 1 - \frac{1}{u} du$$
$$= \pi \Big[ u - ln|u| \Big]_1^{1/2}$$
$$= \pi\left(\tfrac{1}{2} - ln\left(\tfrac{1}{2}\right) - 1 + ln(1)\right)$$
$$= \pi\left(-\tfrac{1}{2} - ln\left(\tfrac{1}{2}\right)\right)$$
$$= 0.6068 \quad \text{units}^3 \quad (4\text{s.f.})$$

**Answer in terms of $\pi$ prefered. Decimal approximation accepted.**

**BM**

Assessor's use only

Question number

3.

b.) i.)

$$\text{RHS} = \int_0^a f(a-x)dx$$

$$= \Big[-F(a-x)\Big]_0^a \qquad \text{where } F(x) = f(x)$$

$$= -F(a-a) + F(a-0)$$

$$= F(a) - F(0)$$

$$\text{LHS} = \int_0^a f(x)dx$$

$$= \Big[-F(x)\Big]_0^a$$

$$= F(a) - F(0) \quad = \text{RHS} \qquad \text{as required}$$

AP

ii)

$$\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\,dx$$

$$\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \sin^n(\pi/2 - x)}\,dx$$

The working is not wrong, but insufficienct evidence has been provided to demonstrate the use of insight and flair in solving complex problems.

$$\frac{a}{a+b}$$

$$= \frac{a}{c}$$

$$= \frac{c-b}{c}$$

$$= 1 - \frac{b}{c}$$

NS

Assessor's use only

Question number

Question Four (a)
(i)

$v = 1 + i \qquad z = x + iy$

$z - v = (x + iy) - (1 + y)$

$= (x - 1) + i(y - 1)$

$vz = (1 + i)(x + iy)$

$= x + iy + ix + i^2 y$

$= (x - y) + i(y + x)$

$\therefore |z - v| =$

$\sqrt{(x-1)^2 + (y-1)^2}$

$= \sqrt{x^2 - 2x + 1 + y^2 - 2y + 1}$

$= \sqrt{x^2 - 2x + y^2 - 2y + 2}$

$\therefore |vz| = \sqrt{(x-y)^2 + (y+x)^2}$

$= \sqrt{x^2 + y^2 - 2xy + y^2 + x^2 + 2xy}$

$= \sqrt{2x^2 + 2y^2 - 4xy}$

$\therefore |vz| = 1z - v1$

Remove $\sqrt{\ }$ by squaring both sides

$\sqrt{2x^2 + 2y^2} = \sqrt{x^2 - 2x + y^2 - 2y + 2}$

$2x^2 + 2y^2 = x^2 - 2x + y^2 - 2y + 2$

$x^2 + y^2 + 2x + 2y = 2$

$(x+1)^2 + (y+1)^2 = 4$

$\therefore$ Equation of circle Radius = 2

^

MEI

**BS**

**Minor error ignored – centre of circle not given.**
**BS grade awarded**

Assessor's use only

Question number

Question Four (a)
(ii)

$|z - v| = \sqrt{x^2 + y^2} = 2x - 2y + 2$

$z + v = x + iy + 1 + i$

$= (x + 1) + i(y + 1)$

$|z + v| = \sqrt{(x + 1)^2 + (y + 1)^2}$

$= \sqrt{x^2 + 2x + 1 + y^2 + 2y + 1}$

$|z - v| = |z + v|$

$\Rightarrow \cancel{x^2} + \cancel{y^2} - 2x - 2y + \cancel{2} = \cancel{x^2} + 2x + \cancel{y^2} + 2x + \cancel{2}$

$-4x + 4y = 0$

$y = x \Rightarrow$ Equation of straight line

MEI

MEI

**$y = -x$ negative sign missing**

Intersection with circle

Circle $= (x + 1)^2 + (y + 1)^2 = 4 \qquad y = x$

$(x + 1)^2 + (x + 1)^2 = 4$

$2x^2 + 4x + 2 = 4$

$2x^2 + 4x - 2 = 0$

$x = 0.4142, -2.414$

$\therefore y = 0.4142, -2.414$

$\therefore$ Points of intersection $=$

$(0.4142, 0.4142)$ and $(-2.414, -2.414)$

**BS**

**Candidate made a minor error which was ignored. Sufficient evidence was provided to show the candidate can select appropriate combinations of techniques and concepts to solve complex problems.**

Assessor's use only

Question number

Question Four (b)

$z^5 = 1cis2n\pi$

$z = \sqrt[5]{1}\, cis\frac{2n\pi^2}{5}$

$= cis\frac{2n\pi}{5}$

$n = 0\ z_1 = cis0 \qquad n = 1\quad z_2 = cis\frac{2n\pi}{5}$

$n = 2\ z_3 = cis\frac{4\pi}{5} \qquad n = 3\quad z_4 = cis\frac{6\pi}{5}$

$n = 4\quad z_5 = cis\frac{8\pi}{5}$

$(z^2 \qquad )\ (z^2 \qquad )$

$z^4$ Roots =

NS

**The candidate has been able to make a start but has not provided an extended chain of reasoning sufficient to meet the requirements of the descriptor.**

Assessor's use only

Question number

5 a)

$r = Ae^{k\theta}$ $\qquad$ $r\cos\theta = x$

$r\sin\theta = y$

$x = Ae^{k\theta}\cos\theta$

$\frac{dx}{d\theta} = kAe^{k\theta}\cos\theta - Ae^{k\theta}\sin\theta$

$y = Ae^{k\theta}\sin\theta$

$\frac{dy}{d\theta} = kAe^{k\theta}\sin\theta + Ae^{k\theta}\cos\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$

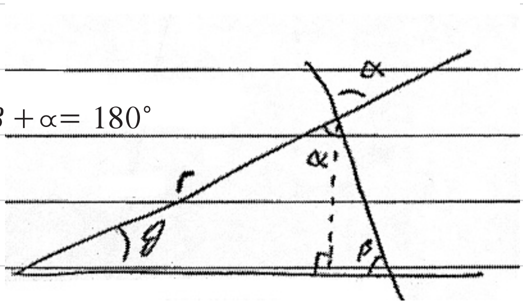
$= \frac{Ae^{k\theta}(k\sin\theta + \cos\theta)}{Ae^{k\theta}(k\cos\theta - \sin\theta)}$

$\frac{dy}{dx} = \frac{k\sin\theta + \cos\theta}{k\cos\theta - \sin\theta}$

BM

b)

$\frac{dy}{dx} = \text{grad} = \frac{\Delta y}{\Delta x}$ $\qquad$ $\theta + B + \alpha = 180°$




$\tan B = \frac{k\sin\theta + \cos\theta}{k\cos\theta - \sin\theta}$

$\ln\frac{r}{A} = \ln e^{k\theta} = k\theta$

N

Assessor's use only

| Question number | |
|---|---|
| 6 a) | $\frac{dx}{dt} = \int 0\,dt$ |
| | $\frac{dx}{dt} = c$ |
| | $x = \int c\,dt$ |
| | $= c_1 t + c_4$ |
| | $\frac{dy}{dt} = \int -g\,dt$ |
| | $\frac{dy}{dt} = -gt + c$ |
| | $y = \int -gt + c_2\,dt$ |
| | $y^2 = -\frac{1}{2}gt^2 + c_2 t + c_3$ |
| | $c_3 = 0,\ c_4 = 0$ because it is originally at the origin |
| | $v^2 = {c_1}^2 + {c_2}^2$ |
| | $y = -\frac{1}{2}g\left(\frac{x}{c_1}\right)^2 + \frac{c_2}{c_2}x$ MEI |
| | $\sin\theta = \frac{c_1}{v}$ |
| | $\tan\theta = \frac{c_1}{c_2}$ |
| | $y = -\frac{1}{2}gx^2\,(\sin\theta v)^{-2} + \tan\theta x$ |

**Although candidate has made several minor errors, all of which have been ignored, there is sufficient evidence to meet the standard.**

**BM**

Assessor's use only

| Question number | |
| --- | --- |
| 6 b) (i) | $y = -\frac{1}{2} g x^2 (v)^2 + \tan\theta x$ |
| | $h = \frac{-g(kh)^2}{zv^2 \sin^2 \ominus} + \tan\theta kh$ |
| | $i = \frac{-gk^2 h}{zv^2 \sin^2 \ominus} + \tan\theta k$ |
| | $zv^2 = -g$ |

candidate has not attempted 6 (b) (ii)

NS

**Candidate has been awarded 2 A's and 8 B's, meeting the requirements for Performance Descriptor 2, Performance Category 3.**